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A Review of Simple Harmonic Motion for Mass Spring System and Its Analogy to the Oscillations in LC Circuit

Getahun Getachew *

Department of Physics, Wolaita Sodo University, Wolaita Sodo, Ethiopia

*Corresponding Author: Getahun Getachew; Email: getahungetachewt@yahoo.com

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ABSTRACT

Simple harmonic motion comes up in many places in physics and provides a generic first approximation to models of oscillatory motion. The mass oscillating on a spring and the oscillation of LC circuit appear to have little in common because the two systems are apparently different systems but both can be described in terms of second order differential equations with constant coefficients of the same form. With the absence of friction in the mass-spring system, the oscillations would continue indefinitely and we obtain equations for the ways in which the displacement, velocity and acceleration of a simple harmonic oscillator vary with time and the ways in which the kinetic and potential energies of the oscillator vary. Similarly, the oscillations of an *LC* circuit with no resistance would continue forever if undisturbed and we obtain equations for time varying charge, current, energy stored in inductor and energy stored in capacitor. Therefore, the aim of this article is to discuss the properties of oscillations of mass-spring system and provide interesting analogies with oscillations in LC circuit.

Keywords: *Mass-spring system, LC circuit, simple harmonic motion, Analogy*

INTRODUCTION

Oscillations occur in many branches of physics, but in this article I discuss the oscillations of mass-spring system without friction and LC circuit oscillations without resistance. A mass oscillating on a spring and the oscillation of LC circuit appear to have little in common; but the mathematics models them is almost indistinguishable and both can be described in terms of a second order differential equations with constant coefficients. In this case the second order differential equation with constant coefficients for equation of motion is

$$\frac{d^2X}{dt^2} = -\omega^2 X \text{--} (1)$$

where, X represents the small displacement (x) from equilibrium position in the mass - spring system or the charge (q) in the LC circuit and ω represents constant coefficients [1, 2]. This equation of motion has a general solution which is given by:

$$X(t) = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t + \phi) \quad - (2)$$

C is called the amplitude of the oscillation, ω is the angular frequency and ϕ is the phase [2].

Furthermore in this article I also briefly describe some of the conditions in which such equations arise and then investigate the reasons why such apparently different systems exhibit very similar behavior.

I. Mass-Spring System

Consider a block of mass *m* attached to the end of a spring, with the block free to move on a horizontal, frictionless surface as shown in the figure below [3].

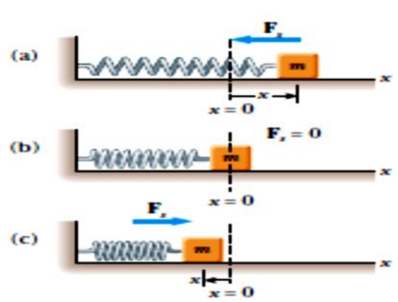

Figure 1: A block attached to a spring moving on a frictionless surface. (a) When the block is displaced to the right of equilibrium ($x > 0$), the force exerted by the spring acts to the left. (b) When the block is at its equilibrium position ($x = 0$), the force exerted by the spring is zero. (c) When the block is displaced to the left of equilibrium ($x < 0$), the force exerted by the spring acts to the right [3] .

When the spring is neither stretched nor compressed, the block is at the position called the equilibrium position of the system, which we identify as $x = 0$. If the block is displaced to a position *x*, the spring exerts a force on the block that is proportional to the position and given by Hooke's law

$$F_s = -kx \text{--} (3)$$

This force is called a restoring force because it is always directed toward the equilibrium position and therefore opposite the displacement from equilibrium.

The system must obey Newton's second law of motion which" states that the force is equal to mass m times acceleration a". Applying Newton's second law $\sum F_x = ma_x$ to the motion of the block, with $F_s = -kx$ providing the net force in the *x* direction, we obtain [2, 3, 4, 5, 6]

$$-kx = ma_x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$a_x = -\frac{k}{m}x \text{--------------------------------------} (5)$$

That is, the acceleration is proportional to the position of the block, and its direction is opposite the direction of the displacement from equilibrium. Systems that behave in this way are said to exhibit simple harmonic motion.

Modeling the block as a particle subject to the force $F_s = -kx$, and assuming that the oscillation occurs along the x-axis, equation (5) can be rewritten as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \text{--------------------------------------} (6)$$

Recall that, by definition, $a_x = \frac{dv_x}{dt}$ and $v_x = \frac{dx}{dt}$ this implies that $a_x = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

If we denote the ratio $\frac{k}{m}$ with the symbol ω^2, $\omega^2 = \frac{k}{m}$

$\omega = \sqrt{\frac{k}{m}}$ -- (7)

Accordingly,

$\frac{d^2x}{dt^2} = -\omega^2 x$-------------------------------------- (8)

The general solution to the above differential equation is

$x(t) = Acos(\omega t + \phi)$------- ------------------ (9)

Where A, ω, and ϕ are constants.

In order to give physical significance to these constants, it is convenient to form a graphical representation of the motion by plotting *x* as a function of *t*, as in figure below.

Figure 2: position versus time graph for an object undergoing simple harmonic motion [4].

A is called the amplitude of the motion, and it is simply the maximum value of the position of the particle in either the positive or negative *x* direction.

The **period** *T* of the motion is the time interval required for the particle to go through one full cycle of its motion.

$T = \frac{2\pi}{\omega}$--- (10)

The inverse of the period is called the frequency *f* of the motion.

$f = \frac{1}{T} = \frac{\omega}{2\pi}$----------------------------------- (11)

The units of *f* are cycles per second, or hertz (Hz). Rearranging the above equation gives

$\omega = 2\pi f = \frac{2\pi}{T}$---------------------------------- (12)

In terms of the characteristics *m* and *k*, the period and frequency of the motion for the particle–spring system can be expressed as follows.

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$------------------------------- (13)

$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$--------------------------------- (14)

Thus the period and frequency depend *only* on the mass of the particle and the force constant of the spring.

The velocity of the particle undergoing simple harmonic motion becomes

$$v_x = \frac{dx}{dt} = A\frac{d}{dt}cos(\omega t + \phi)$$

$v_x = -\omega Asin(\omega t + \phi)$------ ------------------ (15)

$$a_x = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{d(-\omega Asin(\omega t + \phi))}{dt}$$

$a_x = -\omega^2 Acos(\omega t + \phi)$------------------ (16)

Since the sine and cosine functions oscillate between ± 1, the extreme values of the velocity *v* are $\pm\omega A$. Similarly, the extreme values of the acceleration *a* are $\pm\omega^2 A$. Therefore, the maximum values of the magnitudes of the velocity and acceleration are

$v_{max} = \omega A = A\sqrt{\frac{k}{m}}$-----------------------------------(17)

$a_{max} = \omega^2 A = A\frac{k}{m}$-----------------------------------(18)

Graphs of position, velocity and acceleration as a function of time are displayed in real time in the same window, illustrated in Figure 3. As shown in the top and middle plots the maximum and minimum values of the position occur when the velocity is zero. Likewise the maximum and minimum values of velocity occur when the position is at its equilibrium. It is also observed that both graphs position vs. time and velocity vs. time are periodic waves of the same frequency just shifted by 90° or π/2. Furthermore, note that the phase of the acceleration differs from the phase of the position by π radians, or 180° [7].

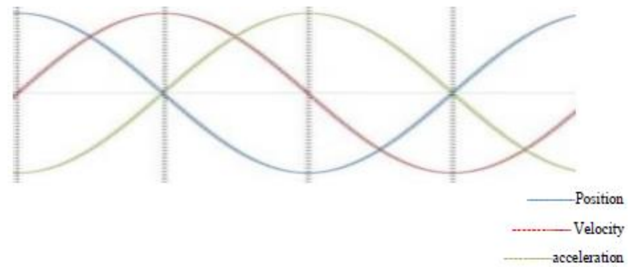

Figure 3: Graphs of position, velocity and acceleration as a function of time [7].

Let us examine the mechanical energy of the block–spring system illustrated in the figure below. The only horizontal force on the block–spring system is the conservative force exerted by an ideal spring. The vertical forces do no work, so the total mechanical energy of the system is conserved [3, 4].

We assume a mass less spring, so the kinetic energy of the system corresponds only to that of the block. Therefore, the kinetic energy of the block is

$K = \frac{1}{2}mv^2$, but $v = -\omega Asin(\omega t + \phi) \rightarrow v^2 = \omega^2 A^2 sin^2(\omega t + \phi)$

$K = \frac{1}{2}m\omega^2 A^2 sin^2(\omega t + \phi)$ --------------------------------------- (19)

The elastic potential energy stored in the spring for any elongation *x* is given by

$U = \frac{1}{2}kx^2$, but $x(t) = Acos(\omega t + \phi) \rightarrow x^2 = A^2cos^2(\omega t + \phi)$

$U = \frac{1}{2}kA^2cos^2(\omega t + \phi)$-- (20)

Since $\omega^2 = \frac{k}{m}$, we can express the total mechanical energy of the simple harmonic oscillator as

$E = K + U = \frac{1}{2}kA^2[sin^2(\omega t + \phi) + cos^2(\omega t + \phi)]$------- (21)

But, from trigonometric identity,

$$sin^2(\omega t + \phi) + cos^2(\omega t + \phi) = 1$$

Therefore, the equation reduces to

$$E = \frac{1}{2}kA^2 \text{----------------------------------- (22)}$$

That is, the total mechanical energy of a simple harmonic oscillator is a constant of the motion and is proportional to the square of the amplitude.

Energy is continuously being transformed between potential energy stored in the spring and kinetic energy of the block. When the body reaches the point $x = A$ its maximum displacement from equilibrium, it momentarily stops as it turns back toward the equilibrium position. That is, when $x = \pm A$, because $v = 0$ at these points the energy is entirely potential, and $E = \frac{1}{2}kA^2$. Because E is constant, it is equal to $E = \frac{1}{2}kA^2$ at any other point. At the equilibrium position (x= 0), the energy is entirely kinetic because U = 0, the total energy, all in the form of kinetic energy is again $E = \frac{1}{2}kA^2$ .We can use the principle of conservation of energy to obtain the velocity for an arbitrary position by expressing the total energy at some arbitrary position x as $E =$

$K + U \rightarrow \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2.$

Solving for v gives

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} = \pm\omega\sqrt{A^2 - x^2} \text{----------------- (23)}$$

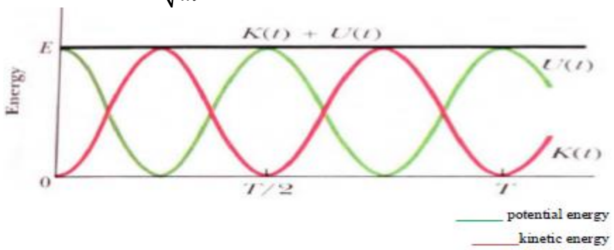

Figure 4: Kinetic energy and potential energy versus time for a simple harmonic oscillator with $\phi = 0$ [8].

II. Oscillations in LC Circuits

LC circuit is the simplest example of an oscillating electrical circuit consists of an inductor L and capacitor C connected together in series with a switch [2]. Unlike a resistor, which always resists the flow of current, an inductor tends to oppose *changes* to the flow of electric current.

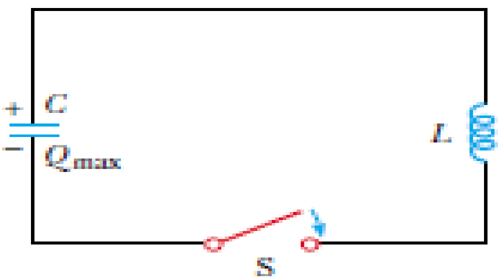

Figure 5: A simple *LC* circuit [3].

A circuit containing an inductor and a capacitor shows an entirely new mode of behavior, characterized by *oscillating* current and charge [2, 3, 4, 5, 6].

The voltage drop v_L across an inductor is given by the formula

$$v_L = L\frac{di}{dt} \text{ -------------------------- (24)}$$

And the voltage drop v_C across a capacitor is given by the formula

$$v_C = \frac{q}{C} \text{ ----------------------------- (25)}$$

To study this oscillation in detail, we apply Kirchhoff's loop rule to the circuit in Fig.5.which states that "the sum of the voltages around any loop of the circuit is zero"

$$v_L + v_C = 0 \text{-------------------------------- (26)}$$

$$-L\frac{di}{dt} - \frac{q}{C} = 0 \text{ ------------------------- (27)}$$

Since $i = \frac{dq}{dt}$ it follows that $\frac{di}{dt} = \frac{d^2q}{dt^2}$

Substituting this expression into equation (27) and dividing by $-L$ one can obtain

$$\frac{d^2q}{dt^2} + \frac{q}{LC} = 0 \text{-------------------------------- (28)}$$

If we denote the $\frac{1}{\sqrt{LC}}$ with the symbol ω, $\omega^2 = \frac{1}{LC}$

$$\omega = \frac{1}{\sqrt{LC}} \text{ --- (29)}$$

Accordingly, $\frac{d^2q}{dt^2} = -\omega^2 q$------------------------ (30)

The Mathematical solution to the above differential equation is

$$q(t) = q_{max}cos(\omega t + \phi) \text{-------- -- (31)}$$

where q_{max}, ω, and ϕ are constants.

The **period** T of the oscillation in LC circuit is

$$T = 2\pi\sqrt{LC} \text{ --- (32)}$$

The inverse of the period is called the frequency f of the oscillation.

$$f = \frac{1}{T} = \frac{1}{2\pi\sqrt{LC}} \text{-- (33)}$$

That is, the period and frequency depend *only* on the charge across the capacitor and the capacitance of the capacitor.

For the given harmonically oscillating charge, the voltage and the current in the LC circuit also oscillate according to eq (25).

$$v = \frac{q_{max}}{C}\cos(\omega t + \phi)$$

$$v = v_{max}\cos(\omega t + \phi) \text{ -- (34)}$$

$$i = \frac{dq(t)}{dt} = q_{max}\frac{d}{dt}cos(\omega t + \phi)$$

$$i = -\omega q_{max}sin(\omega t + \phi) \text{-- (35)}$$

Where ωq_{max} *is* maximum current and is given by

$$i_{max} = \omega q_{max} = q_{max}\frac{1}{\sqrt{LC}} \text{ --(36)}$$

Roughly speaking, if we assume that the capacitor begins charged, then the capacitor begins by discharging through the inductor, slowly at first but picking up speed as the inductor lets more current through. Once the capacitor is fully discharged, the inductor continues pushing current through the circuit, which drains even more charge from the capacitor, leaving it with a negative total charge. The capacitor then reverses the flow of current to regain the lost charge, but the same thing happens again, with the inductor continuing to push current through in the reverse direction until the capacitor is back to its initial charged state. The cycle thus continues indefinitely.

Thus the charge and current in an *L-C* circuit oscillate sinusoidally with time, with an angular frequency determined by the values of L and C.

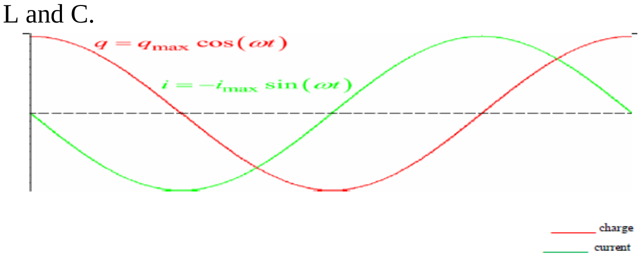

Figure 6: Graphs of charge versus time and current versus time for a résistance less, non radiating *LC* circuit [9].

Since the *L-C* circuit is a conservative system, we can analyze the circuit using an energy approach and we expect that the total energy stored in the system to be constant. When the capacitor is fully charged, the energy *U* in the circuit is stored in the electric field of the capacitor and is equal to $\frac{q_{max}^2}{2C}$. At this time, the current in the circuit is zero, and therefore no energy is stored in the inductor. After the switch is closed, the rate at which charges leave or enter the capacitor plates (which is also the rate at which the charge on the capacitor changes) is equal to the current in the circuit. As the capacitor begins to discharge after the switch is closed, the energy stored in its electric field decreases. The discharge of the capacitor represents a current in the circuit, and hence some energy is now stored in the magnetic field of the inductor [3, 4].

The electric potential energy stored in the capacitor is given by

$$U_C = \frac{{q_{max}}^2}{2C}cos^2(\omega t + \phi) \text{------------- (37)}$$

The magnetic energy stored in the inductor,

$$U_L = \frac{Li^2}{2}sin^2(\omega t + \phi) \text{------------- - (38)}$$

Combining equations (37) and (38) the total electrical energy of the LC oscillator can be expressed as

$$U = U_C + U_L = \frac{1}{2C} q_{max}{}^2 [sin^2(\omega t + \phi) + cos^2(\omega t + \phi)] \quad \text{--- (39)}$$

But, from trigonometric identity,

$$sin^2(\omega t + \phi) + cos^2(\omega t + \phi) = 1$$

Therefore, the equation reduces to

$$U = \frac{1}{2C} q_{max}{}^2 \text{---------------- (40)}$$

That is, the total electrical energy of a LC oscillator is a constant of the motion and is proportional to the square of the amplitude charge. Energy is continuously being transformed between potential energy stored in the capacitor and inductor. We can use the principle of conservation of energy to obtain the expression for electric current as

$$U = U_C + U_L = \rightarrow \frac{q^2}{2C} + \frac{Li^2}{2} = \frac{1}{2C} q_{max}{}^2$$

Solving for i gives

$$i = \pm\sqrt{\frac{1}{LC}(q_{max}{}^2 - q^2)} = \pm\omega\sqrt{q_{max}{}^2 - q^2} \text{------- (41)}$$

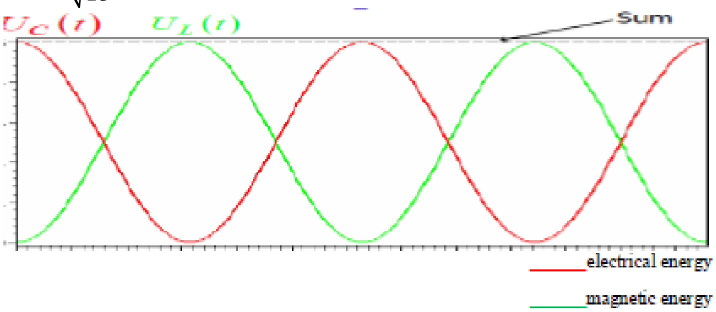

Figure.7: Graphs of magnetic energy stored in inductor and electric potential energy stored in capacitor versus time for oscillations in LC Circuit with $\phi = 0$ [9].

III. Comparing Simple harmonic motion for a mass-spring system with the oscillations in LC Circuit

- Comparing equations (8-23) with equations (30-41), the simple harmonic motion for a mass-spring system and oscillations in LC Circuit have the same form. Thus all the discussions about the simple harmonic motion for a mass-spring system can be carried over to oscillations in LC Circuit. Moreover one can see a direct correspondence between the two sets of physical quantities involved:
  - Displacement (x) corresponds to the charge(q);
  - Velocity (v_x) corresponds to the current(i);
  - Inductance(L) corresponds to the mass (m);
  - Spring constant (k) corresponds to the Inverse of capacitance ($1/C$).
  - Force(F_x) corresponds to the voltage(v) ;
  - Kinetic energy of moving block corresponds to the magnetic energy stored in inductor and
  - Potential energy stored in a stretched spring corresponds to the electric potential energy stored in the capacitor.
- Comparing graphs in figure (3) with figure (6), for oscillating mass spring system the maximum and minimum values of velocity occur when the displacement is zero and the maximum and minimum values of displacement occur when the velocity is zero; and in LC circuit oscillation, the maximum and minimum values of current occur when the charge on the capacitor is zero and the maximum and minimum values of charge occur when the current in the circuit is zero. Thus the displacement of a stretched spring is analogous to the charge on the capacitor and the velocity of the moving block of mass m is analogous to the current in the inductor.
- Comparing graphs in figure (4) with figure (7), for oscillating mass spring system, the total energy is equal to the maximum potential energy stored in the spring when $x = \pm A$, and at x= 0 the total energy is equal to the kinetic energy and is equal to $E = \frac{1}{2}kA^2$; and in LC circuit oscillation, the total energy is equal to maximum electric potential energy stored in the electric field of the capacitor when the current in the circuit is zero and at q= 0, the total energy is all in the form the magnetic energy stored in the inductor and is equal to $U = \frac{q_{max}^2}{2C}{}^2$. This implies that the potential energy stored in a stretched spring is analogous to the electric potential energy stored in the capacitor and the kinetic energy of the moving block is analogous to the magnetic energy stored in the inductor.

Therefore, the properties oscillations of mass-spring system are very important and provide interesting analogies with oscillations in LC circuit.

The following table summarizes the physical quantities involved in simple harmonic motion for mass-spring system and L-C circuit oscillations and the analogies between them.

Mass-spring system		LC circuit	
Quantities	Equations	Quantities	Equations
Position(x)	$x(t) = Acos(\omega t + \phi)$	Charge	$q(t) = q_{max}cos(\omega t + \phi)$
Velocity(v_x)	$v_x(t) = -\omega Asin(\omega t + \phi)$	Current	$i(t) = -\omega q_{max}sin(\omega t + \phi)$
Acceleration (a_x)	$a_x(t) = -\omega^2 Acos(\omega t + \phi)$	Rate of change of current	$\frac{di(t)}{dt} = -\omega^2 q_{max}cos(\omega t + \phi)$
Force (F_x)	$F_x(t) = -kAcos(\omega t + \phi)$	Voltage	$v(t) = \frac{q_{max}}{C}cos(\omega t + \phi)$
Mass(m)	$m = \frac{k}{\omega^2}$	Inductance	$L = \frac{1}{C\omega^2}$
$\frac{1}{k}$(k=spring constant)	$\frac{1}{k} = \frac{1}{m\omega^2}$	Capacitance	$C = \frac{1}{L\omega^2}$
Angular frequency	$\omega = \sqrt{\frac{k}{m}}$	Angular frequency	$\omega = \sqrt{\frac{1}{LC}}$
Period	$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$	Period	$T = 2\pi\sqrt{LC}$

Frequency	$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$	Frequency	$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$
Kinetic energy of moving block	$K = \frac{1}{2}mv_{max}^2 sin^2(\omega t + \phi)$	Magnetic energy stored in inductor	$U_L = \frac{1}{2}Li_{max}^2 sin^2(\omega t + \phi)$
Potential energy stored in spring	$U = \frac{1}{2}kA^2 cos^2(\omega t + \phi)$	Electric energy stored in capacitor	$U_C = \frac{1}{2C}{q_{max}}^2 cos^2(\omega t + \phi)$
Total energy	$E = \frac{1}{2}kA^2$	Total energy	$U = \frac{1}{2C}{q_{max}}^2$

CONCLUSION

Generally speaking, the mass-spring system and the LC circuit are the two very different physical systems but both can be described by similar second order differential equations with constant coefficients because mathematical model is the same in both cases. Comparing equation (8) with equation (30), the two equations have the same form and the general solutions for displacement and charge take the same form as indicated in equations (9) and (31). This shows that the displacement oscillation of the mass-spring system driven by an externally supplied sinusoidal force is analogous to the charge oscillation in the LC circuit driven by an externally supplied sinusoidal voltage. Thus the similar discussions can be carried over the simple harmonic motion for a mass-spring and oscillations in LC Circuit. Moreover there is a direct correspondence between charge (q) and displacement (x); current (i) and velocity (v_x) ; inductance (L) and mass (m); inverse of capacitance ($1/C$) and spring constant (k) ;force(F_x) and voltage(v) ; kinetic energy of moving block and magnetic energy stored in inductor; potential energy stored in a stretched spring and electric potential energy stored in the capacitor. Therefore, the properties of oscillations of mass-spring system are very important and provide interesting analogies with oscillations in LC circuit.

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